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Uniform exponential stability approximations of semi-discretization schemes for two hybrid systems

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This paper deals with the uniform exponential stabilities (UESs) of two hybrid control systems consisting of wave equation and a second-order ordinary differential equation. Linear feedback law and local viscosity, and nonlinear feedback law and interior anti-damping are considered, respectively. Firstly, the hybrid system is reduced to a first order port-Hamiltonian system with dynamical boundary conditions and the resulting systems are then discretized by average central-difference scheme. Secondly, the UES of the discrete system is obtained without prior knowledge on the exponential stability of continuous system. The frequency domain characterization of UES for a family of contractive semigroups and discrete multiplier method are utilized to verify main results, respectively. Finally, the convergence analysis of the numerical approximation scheme is performed by the Trotter-Kato Theorem. Most interestingly, the exponential stability of the continuous system is derived by the convergence of energy and UES and this

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is a new idea to investigate the exponential stability of some complicate systems. The effectiveness of the numerical approximating scheme is verified by numerical simulation.

Keywords: Hybrid system; nonlinear feedback; anti-damping; exponential stability; finite difference.

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1. Introduction

The wave equation on a segment with dynamical boundary control and local viscosity damping is a simple hybrid system, which is a mixed set of partial differential equation (PDE) and ordinary differential equation (ODE). More precisely, if let $x \in (0,1)$ be spacial variable, and $t \geq 0$ the time variable, the hybrid system is described by

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t) - a(x)\partial_t u(x,t), \tag{1.1}$$

$$m\partial_{tt}u(0,t) - \partial_x u(0,t) = \mathcal{F}(t), \tag{1.2}$$

where, $m > 0$ is the tip mass, and for simplicity, the wave speed and the length of the segment are chosen to be unity. $a(x)$ and $\mathcal{F}(t)$ are real scalar functions satisfying some properties which will be specified later and $\mathcal{F}(t)$ is the control at the boundary $x = 0$. It is well known that (1.1) and (1.2) can be used to describe both the vibration cable with a tip mass attached at the free end (e.g. see Ref. 21, Ref. 6, Ref. 20) and the mechanical oscillation inside the drill pipe (e.g. see Ref. 26, Ref. 25).

In the case of $a(x) = 0$, designing suitable and effective feedback control laws to stabilize (1.1) and (1.2) were main contributions of the works mentioned above. For instance, Morgul considered the clamped boundary $u(t,1) = 0$ at one extreme and linear feedback control law $\mathcal{F}(t) = -au_t(0,t) + \alpha u_{xt}(0,t)$ with positive a and α at another extreme. The resulting closed-loop system is

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t), \tag{1.3}$$

$$u(t,1) = 0, \tag{1.4}$$

$$m\partial_{tt}u(0,t) - \partial_x u(0,t) = -au_t(0,t) + \alpha u_{xt}(0,t). \tag{1.5}$$

The exponential stability of (1.3)-(1.5) was lengthly discussed in Ref. 21. Moreover, Guo and Xu determined the optimal decay rate of the energy for a case left unsolved in Ref. 21, and the asymptotic expansion of the associated semigroup was also obtained [6]. Recently, Vansprangh, Ferrante, and Prieur designed a class of nonlinear stabilizing feedback that only depends on the velocity at the controlled extreme. More precisely, the nonlinear feedback law $\mathcal{F}(t) = F(u_t(0,t))$ and nonlinear dissipative Neumann velocity feedback $u_x(1,t) = -g(u_t(1,t))$ were incorporated into

(1.1)-(1.2) and the corresponding closed-loop system was

$$\partial_{tt}u(x,t) = \partial_{xx}u(x,t), \tag{1.6}$$

$$m\partial_{tt}u(0,t) - \partial_x u(0,t) = F(u_t(0,t)), \tag{1.7}$$

$$u_x(1,t) = -g(u_t(1,t)). \tag{1.8}$$

The exponential decay of the above system was investigated under different assumptions on the functions F and g in Ref. 25. The stabilization and regulation of a variant of hybrid system (1.1) and (1.2) was studied by using a proportional integral boundary controller in Ref. 26. However, the system of Ref. 26 is linear but the elasticity of the propagation medium is allowed to be non-homogenous. Furthermore, if external disturbance was taken into consideration and entered the system through (1.2), Mei constructed an infinite-dimensional extended state observer to estimate the total disturbance and state simultaneously. An estimated state based controller was then designed to stabilize the hybrid system (1.1) and (1.2) [20].

Table 1. Known Results for Uniform Exponential Stability

System	Discrete scheme	Verification method	Y/N
Wave	Finite difference (FD)	Spectral analysis[9,24]	N
Wave	Finite difference (FD)	Numerical experiment[2]	N
Wave	Finite element (FM)	Spectral analysis[9]	N
Wave	Finite element (FM)	Numerical experiment[2]	N
Wave	Mixed finite element	Matrix inequality[2]	Y
Wave	Mixed finite element	Lyapunov function[5]	Y
Wave	FD with viscosity	discrete multiplier[23,24]	Y
wave	FD with order reduction	Lyapunov function[13,30]	Y
Thermoelasticity	FM	Frequency domain[16]	Y
Heat-wave coupling	FD with order reduction	Lyapunov function[29]	Y
Schrodinger on $H^1(0,1)$	FD with order reduction	Lyapunov function[15]	Y
Schrodinger on $L^2(0,1)$	FD with order reduction	Frequency domain[7]	Y
Eulur beam	Finite volume	Lyapunov function[14]	Y
Timoshenko beam	FD with order reduction	Frequency domain[27]	Y

In this paper, we continue to investigate the hybrid system (1.1) and (1.2) from the viewpoint of numerical approximation. That is to say we first need to perform numerical discretization for the component of PDE and transform it into a family of ODEs. A family of semi-discretized systems, which consists of pure ODEs, is derived from hybrid system mentioned above. We want to study the UES of the discrete systems, i.e., whether or not the discrete energies uniformly exponentially decay with respect to the mesh size. To draw a conclusion for this problem, one

should face three challenges. The first one is which approximating scheme is chosen among many approximating methods to discretize the spatial variable. The second one is which method is used to verify the UES. The last one is how to verify the solution of semi-discretized system converges to the solution of the original continuous system. Though one may can't obtain uniform decay by using the classical finite difference and finite element schemes since these semi-discretizations produce spurious oscillations of the high frequencies [24,31]. The first question and the third question are easy to cope with because there are rich results on numerical solution to PDE [12]. The second challenge is the most difficult one. However, some successful pioneering works have been given in last two decades. We refer to these references and collect some related results in the table 1.

The natural idea of verifying the UES is to follow each step of continuous case (e.g. see Ref. 5, Ref. 13-Ref. 15 and Ref. 24-Ref. 25). Whereas the method for the exponential stability of the continuous may not suit for the discrete case or vice versa. Therefore, it is necessary to obtain UES without prior knowledge on how to verify the exponential stability of continuous system. On the contrary, the exponential stability of the original continuous system is obtained by UES of discrete systems and convergence analysis (see Theorem 2.4 and Theorem 2.5). This not only can greatly enlarge the study scope of UES but also conversely research the exponential stability of the continuous system. For instance, to the best of our knowledge, the famous Huang-Prüss's characteristic condition of exponential stability (e.g. see Ref. 16, Ref. 1, and Section 8.1 of Ref. 11) is difficult to be appled for the hybrid system with linear feedback law discussed in this note. But it's discrete version is effective in the UES of semi-discretization systems (see the proof of Theorem 3.1 below). In that case, we investigate the hybrid system with nonlinear feedback law and anti-damping by similar idea. Because the absence of characteristic condition of exponential stability for nonlinear semigroup, we discuss UES by using the discrete multiplier method. Both boundary and in-domain anti-damping, which are more interesting in terms of applications and were widely discussed,[3,22,4,8,19,28,26] are considered in our model.

Thus, the contribution of this work is three-fold:

- Extend the result of uniform exponential stability from PDE with linear feedback and boundary damping to the PDE with nonlinear feedback and anti-damping.
- The characteristic condition of uniform exponential stability for a family of semigroups of contraction is successfully applied to the discrete system of boundary control hybrid system. This method has potential applications for the system in which the Lyapunov function is absent.
- Present a new idea to discuss the exponential stability of distributed parameter system, i.e., the uniform exponential stability and convergence of the energy implies that the continuous system is exponential stable (see Remark 2.1).

The structure of this paper is organized as follows. In section 2, we study the hybrid system consisting of (1.1) and (1.4) with linear feedback (1.5). A suitable semi-discretization system, which is uniformly exponentially stable with respect to the discretized parameter and convergent to the continuous system as mesh size tends to zero, is built. The exponential stability of the continuous hybrid system are also discussed in this section. In section 3, similar results of the second hybrid system consisting of (1.1) (1.7), and (1.8) are obtained. The effectiveness of the numerical approximating algorithms are also verified by numerical simulations in this section. In section 4, concluding remarks and future directions are shown.

2. The hybrid system with linear feedback

In this section, we present main results on the hybrid system (1.1), (1.4), and (1.5). Firstly, we transform them into a first order port-Hamiltonian system with dynamical boundary condition, which is a natural treatment for the wave equation (e.g. see Ref. 19 and Section 7.1 of Ref. 11), and discretize the resulting system by average central-difference method. Secondly, the UES of the discrete systems is proved by the frequency domain method. Thirdly, the stability and the consistence of numerical approximating scheme are presented by using the Trotter-Kato Theorem. In the sequel, we denote A^* in stead of $A^\top$ the transpose of a matrix A. The prime $'$ denotes a differential with respect to the temporal variable. Moreover, let $N \in \mathbb{N}$ be a positive integer and $h = 1/(N+1)$ step size, the index k and j are fixed to denote $k = 0, 1, \cdots, N+1$ and $j = 0, 1, \cdots, N$ throughout the whole paper. In this section, $a(x)$ is a bounded nonnegative continuous function on $(0,1)$, i.e., $a(x) > 0$ for $x \in \omega$ and $a(x) = 0$ for $x \in (0,1)\setminus\omega$ with $\omega \subset (0,1)$ being a nonempty open set.

2.1. *Order reduction and discretization*

Firstly, we define the following intermediate variables $v(x,t) = \partial_t u(x,t),\ w(x,t) = \partial_x u(x,t)$. Then, the hybrid system consisting of (1.1), (1.4), and (1.5) can be rewritten as

$$\partial_t v(x,t) = \partial_x w(x,t) - a(x)v(t,x), \tag{2.1}$$

$$\partial_t w(x,t) = \partial_x v(x,t), \tag{2.2}$$

$$\eta'(t) = w(0,t) - av(0,t), \tag{2.3}$$

$$v(1,t) = 0, \tag{2.4}$$

$$v(x,0) = v^0(x), \quad w(x,0) = w^0(x), \tag{2.5}$$

in which $\eta(t) = mv(0,t) - \alpha w(0,t)$, $v^0(x)$ and $w^0(x)$ are the initial velocity and the initial force of the elastic cable. Define the state space of (2.1)-(2.5) to be $\mathbb{X} = L^2((0,1);\mathbb{C}^2) \times \mathbb{C}$ with the inner product

$$\langle Y, \widetilde{Y}\rangle_{\mathbb{X}} = \langle v, \widetilde{v}\rangle_{L^2} + \langle w, \widetilde{w}\rangle_{L^2} + \frac{1}{m + a\alpha}\eta\overline{\widetilde{\eta}}$$

for $Y = (v, w, \eta)$, $\widetilde{Y} = (\widetilde{v}, \widetilde{w}, \widetilde{\eta}) \in \mathbb{X}$ and $\langle\cdot,\cdot\rangle_{L^2}$ is canonical inner product on $L^2(0,1)$. Now we can formulate (2.1)-(2.5) into the abstract Cauchy problem

$$Y'(t) = AY(t), \quad Y(0) = (v^0, w^0, \eta_0)^* \in \mathbb{X}, \tag{2.6}$$

here, $Y(t) = (v(x,t), w(x,t), \eta(t))^*$, $\eta_0 = \eta(0)$, and the operator A is defined by

$$AY(t) = \begin{pmatrix} w_x(t,x) - a(x)v(t,x) \\ v_x(t,x) \\ w(0,t) - av(0,t) \end{pmatrix} \tag{2.7}$$

and

$$D(A) = \left\{(v, w, \eta)^* \in H^1((0,1);\mathbb{C}^2) \times \mathbb{C} : v(1) = 0,\ \eta = mv(0) - \alpha w(0)\right\}. \tag{2.8}$$

Using Lumer-Phillips Theorem, we can obtain the well-posedness result of (2.6). The proof is omitted since it is similar to that of Theorem 1 in Ref. 21.

Theorem 2.1. *The operator $A:\ D(A) \to \mathbb{X}$ defined by (2.7) and (2.8) generates a C_0 semigroup of contraction $T(t)$ on the state space.*

In fact, it is easy to see that for any $Y \in D(A)$, we have

$$\begin{aligned} &2\mathrm{Re}\langle AY, Y\rangle_{\mathbb{X}} = \langle AY, Y\rangle_{\mathbb{X}} + \langle Y, AY\rangle_{\mathbb{X}} \\ &= \int_0^1 (\overline{v}w)_x + (\overline{w}v)_x dx - 2\int_\omega a(x)|v(x)|^2 dx + \frac{\overline{\eta}(w(0) - \alpha v(0)) + \eta\overline{(w(0) - \alpha v(0))}}{m + a\alpha}, \end{aligned}$$

which implies that

$$\mathrm{Re}\langle AY, Y\rangle_{\mathbb{X}} = -\frac{m\alpha|v(0)|^2 + a|w(0)|^2}{m + a\alpha} - \int_\omega a(x)|v(x)|^2 dx. \tag{2.9}$$

However, it is well known that (2.9) has tight relation with the energy functional $E(t)$ of (2.1)-(2.5). In other words, we define $E(t)$ of (2.1)-(2.5) by

$$E(t) = \frac{1}{2}\int_0^1 |v(t,x)|^2 + |w(t,x)|^2 dx + \frac{|\eta(t)|^2}{2(m + a\alpha)} \tag{2.10}$$

and differentiate it along the solutions of (2.1)-(2.5) to obtain

$$\begin{aligned} E'(t) &= \mathrm{Re}\langle AY(t), Y(t)\rangle_X \\ &= -\int_\omega a(x)|v(x,t)|^2 dx - \frac{1}{m + a\alpha}[m\alpha|v(0,t)|^2 + a|w(0,t)|^2]. \end{aligned} \tag{2.11}$$

Remark 2.1. Up to now, we don't know whether or not the system (2.1)-(2.5) is exponentially stable. It is not easy to utilize the Lyapunov function methods of the Ref. 5, Ref. 30, and Ref. 21 and frequency domain method of the Ref. 1, Ref. 16, and Ref. 17 to verify this property. We could discuss the UES of (2.1)-(2.5) without prior knowledge of the exponential stability of continuous system. However, the UES and convergence of the energies between the discrete system and continuous implies that

the continuous system is exponential stable. This is one of main contributions this paper.

Secondly, we introduce the so called average central-difference semi discretization scheme for (2.1)-(2.5). To this end, insert $N+2$ points and $N+1$ points, denoted by $x_k = kh$ and $y_j = (j+1/2)h$ respectively, in the domain $[0,1]$. The notations

$$\delta_x f_{j+\frac{1}{2}} = \frac{f_{j+1} - f_j}{h}, \quad f_{j+\frac{1}{2}} = \frac{f_{j+1} + f_j}{2}$$

denote the central-difference operator of $f_x(x)$ and the average operator of $f(x)$ at the node y_j, respectively. If x in (2.1) and (2.2) is replaced by y_j, then we have

$$\partial_t v(y_j,t) = \partial_x w(y_j,t) - a(y_j)v(y_j,t),$$
$$\partial_t w(y_j,t) = \partial_x v(y_j,t).$$

In the above identities, approximating the temporal derivatives of the left-hand side by the average operators and the spacial derivatives of right-hand side the central-difference operators respectively, one derives

$$v'_{j+\frac{1}{2}}(t) = \delta_x w_{j+\frac{1}{2}}(t) - a_j v_{j+\frac{1}{2}}(t), \tag{2.12}$$
$$w'_{j+\frac{1}{2}}(t) = \delta_x v_{j+\frac{1}{2}}(t), \tag{2.13}$$
$$\eta'(t) = w_0(t) - a v_0(t), \tag{2.14}$$
$$\eta(t) = m v_0(t) - \alpha w_0(t), \tag{2.15}$$
$$v_{N+1}(t) = 0, \tag{2.16}$$
$$v_k(0) = v_k^0, \quad w_k(0) = w_k^0, \tag{2.17}$$

where $v_k(t)$ and $w_k(t)$ are grid functions at grids x_k, and v_k^0 and w_k^0 the approximations of the initial values $v^0(x_k)$ and $w^0(x_k)$, respectively. Notice that the viscosity terms $a(y_j)v(y_j,t)$ are in fact approximated by $a(x_j)[v(x_{j+1},t)+v(x_j,t)]/2$. Please reader to Ref. 5, Ref. 13, and Ref. 30 for more information.

Take the state spaces of (2.12)-(2.17) to be

$$\mathbb{X}_h = \mathbb{C}^{N+1} \times \mathbb{C}^{N+1} \times \mathbb{C}.$$

Assume that $Y_h = (v,w,\eta) \in \mathbb{X}_h$ with $v = (v_0, v_1, \cdots, v_N)$ and $w = (w_1, w_2, \cdots, w_{N+1})$. If setting $w_0 = \alpha^{-1}(m v_0 - \eta)$ in light of (2.15), then we can define the inner product of $\mathbb{X}_h$ by

$$\left\langle Y_h, \widetilde{Y}_h \right\rangle_{\mathbb{X}_h} = h \sum_{j=0}^{N} \left[v_{j+\frac{1}{2}} \overline{\widetilde{v}_{j+\frac{1}{2}}} + w_{j+\frac{1}{2}} \overline{\widetilde{w}_{j+\frac{1}{2}}} \right] + \frac{\eta \overline{\widetilde{\eta}}}{m + a\alpha},$$

for $Y_h = (v,w,\eta)$, $\widetilde{Y}_h = (\widetilde{v}, \widetilde{w}, \widetilde{\eta}) \in \mathbb{X}_h$.

Let $Y_h(t) = (v_h(t), w_h(t), \eta(t))^*$ with

$$v_h(t) = (v_0(t), v_1(t), \cdots, v_N(t)), \tag{2.18}$$
$$w_h(t) = (w_1(t), w_2(t), \cdots, w_{N+1}(t)), \tag{2.19}$$

being unknown variables of (2.12)-(2.17). The energy functional, which is denoted by $E_h(t)$, of (2.12)-(2.17) is defined by

$$E_h(t) = \frac{1}{2}\langle Y_h(t), Y_h(t)\rangle_h = \frac{h}{2}\sum_{j=0}^{N}\left[\left|v_{j+\frac{1}{2}}(t)\right|^2 + \left|w_{j+\frac{1}{2}}(t)\right|^2\right] + \frac{|\eta(t)|^2}{2(m+a\alpha)}. \quad (2.20)$$

Finally, $E_h(t)$ is non-increasing and this is parallel with (2.11). More precisely, we have

Proposition 2.1. *The discrete energy functional $E_h(t)$ satisfies*

$$E_h'(t) = -h\sum_{jh\in\omega} a_j\left|v_{j+\frac{1}{2}}(t)\right|^2 - \frac{1}{m+a\alpha}[m\alpha|v_0(t)|^2 + a|w_0(t)|^2]. \quad (2.21)$$

Proof. Multiplying (2.12), (2.13), and (2.14) by $hv_{j+\frac{1}{2}}(t)$, $hw_{j+\frac{1}{2}}(t)$, and $\eta(t)/(m+a\alpha)$ respectively and summing for j from 0 to N, we derive

$$\begin{aligned} E_h'(t) = &-\sum_{jh\in\omega} a_j\left|v_{j+\frac{1}{2}}(t)\right|^2 \\ &+ \frac{\eta(t)\eta'(t)}{m+a\alpha} + h\sum_{j=0}^{N}\left[v_{j+\frac{1}{2}}(t)\delta_x w_{j+\frac{1}{2}}(t) + w_{j+\frac{1}{2}}(t)\delta_x v_{j+\frac{1}{2}}(t)\right]. \end{aligned} \quad (2.22)$$

However, by some simple calculations, we have

$$\begin{aligned} &h\sum_{j=0}^{N}\left[v_{j+\frac{1}{2}}(t)\delta_x w_{j+\frac{1}{2}}(t) + w_{j+\frac{1}{2}}(t)\delta_x v_{j+\frac{1}{2}}(t)\right] \\ =&\sum_{j=0}^{N}\frac{(v_{j+1}(t)+v_j(t))(w_{j+1}(t)-w_j(t))}{2} + \sum_{j=0}^{N}\frac{(w_{j+1}(t)+w_j(t))(v_{j+1}(t)-v_j(t))}{2} \\ =&v_{N+1}(t)w_{N+1}(t) - v_0(t)w_0(t). \end{aligned} \quad (2.23)$$

Thus, (2.14)-(2.16) and (2.22)-(2.23) implies that (2.21) holds. □

2.2. *Uniform exponential stability of (2.12)-(2.17)*

To present main result of this subsection, we first rewrite (2.12)-(2.17) as state space formulation. For this purpose, let $e_1 = (1, 0, \cdots, 0) \in \mathbb{C}^{(N+1)}$, and

$$B_{N+1} = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \quad M_{N+1} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix},$$

$\Lambda_{N+1} = \mathrm{diag}\{1, 0, \cdots, 0\}$, $\Lambda_a = \mathrm{diag}\{a_0, a_1, \cdots, a_N\}$ belong to $\mathbb{C}^{(N+1)\times(N+1)}$. Set

$$B_h = \frac{1}{2}\begin{pmatrix} B_{N+1} & 0 & 0 \\ 2m\alpha^{-1}\Lambda_{N+1} & B_{N+1}^* & -2\alpha^{-1}e_1^* \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$M_h = \frac{1}{h}\begin{pmatrix} -m\alpha^{-1}\Lambda_{N+1} - 2^{-1}h\Lambda_a B_{N+1} & M_{N+1} & \alpha^{-1}e_1^* \\ -M_{N+1}^* & 0 & 0 \\ h(\alpha^{-1}m - a)e_1 & 0 & -h\alpha^{-1} \end{pmatrix}.$$

Because B_h is evidently invertible, (2.12)-(2.17) are then equivalent to the differential equation

$$B_h Y_h'(t) = M_h Y_h(t) \Leftrightarrow Y_h'(t) = A_h Y_h(t),\ Y_h(0) \in \mathbb{X}_h,\ \text{with } A_h := B_h^{-1}M_h. \quad (2.24)$$

It is easy to see that B_h is corresponding to the average operator of time derivative of (2.12)-(2.13). If one replace B_h by identity operator, then the classical finite difference scheme of (2.1)-(2.5) is easily restored from (2.24), i.e.,

$$\dot{Y}_h(t) = M_h Y_h(t), \quad Y_h(0) \in \mathbb{X}_h.$$

Here we explain the significance of the discrete scheme (2.24). We plot two figures in Figure 1- Figure 2, respectively. Figure 1 depicts the maximal real parts of A_h and M_h for $N = 40 : 5 : 400$. Figure 2 depicts the distributions of the eigenvalues of A_h and M_h in which $N = 500$.

We see that the real parts of the eigenvalues of M_h approach to zero and those of A_h approach to a negative number from these figures. In these figures, we take $m = a = 1,\ \alpha = 2$ and $a(x) = \cos((\pi x)/2)$. Numerical simulation results show that the classical finite difference scheme of (2.1)-(2.5) is not uniformly exponentially stable. This conclusion is consistent with earlier research results of Ref. 24. However, Figure 1 manifests (2.24) is perhaps uniformly exponentially stable. In the remaining part of this subsection, we give strict proof about uniform exponential stability of (2.24).

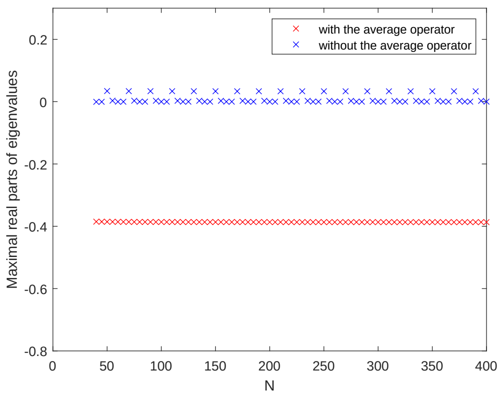

Fig. 1. Maximal real parts of eigenvalues

On the one hand, the inner product on the state space $\mathbb{X}_h$ can be rewritten as

$$\left\langle Y_h, \widetilde{Y}_h \right\rangle_{\mathbb{X}_h} = h \left\langle B_h Y_h,\ B_h \widetilde{Y}_h \right\rangle$$

for $Y_h = (v, w, \eta)$, $\widetilde{Y}_h = (\widetilde{v}, \widetilde{w}, \widetilde{\eta}) \in \mathbb{X}_h$. Here $\langle \cdot, \cdot \rangle$ are canonical inner product of $\mathbb{C}^{2N+3}$ and $w_0 := \alpha^{-1}(mv_0 - \eta)$ is applied in the last identity. On the other hand, $\forall h \in (0, 1)$, the operator A_h, which is discrete counterpart of A, are dissipative on the space $\mathbb{X}_h$. Because the dissipativity of A_h is key in the next part, we put it in the following property.

Proposition 2.2. *For all $0 < h < 1$, the operator A_h is dissipative and generates C_0-semigroup of contractions T_h on the space $\mathbb{X}_h$.*

Proof. For any $Y_h = (v, w, \eta) \in \mathbb{X}_h$, we consider

$$\langle A_h Y_h, Y_h \rangle_{\mathbb{X}_h} = \langle M_h Y_h,\ B_h Y_h \rangle$$
$$= -h \sum_{jh \in \omega} a_j \left| v_{j+\frac{1}{2}} \right|^2 + h \sum_{j=0}^{N} \left[\overline{v_{j+\frac{1}{2}}} \delta_x w_{j+\frac{1}{2}} + \overline{w_{j+\frac{1}{2}}} \delta_x v_{j+\frac{1}{2}} \right] + \frac{\overline{\eta}[(\alpha^{-1}m - a)v_0 - \alpha^{-1}\eta]}{m + a\alpha},$$

and

$$\langle Y_h, A_h Y_h \rangle_{\mathbb{X}_h} = \langle B_h Y_h,\ M_h Y_h \rangle$$
$$= -h \sum_{jh \in \omega} a_j \left| v_{j+\frac{1}{2}} \right|^2 + h \sum_{j=0}^{N} \left[v_{j+\frac{1}{2}} \overline{\delta_x w_{j+\frac{1}{2}}} + w_{j+\frac{1}{2}} \overline{\delta_x v_{j+\frac{1}{2}}} \right] + \frac{\eta \overline{[(\alpha^{-1}m - a)v_0 - \alpha^{-1}\eta]}}{m + a\alpha}.$$

By using $\eta = mv_0 - \alpha w_0$ and some simple operations, we obtain

$$\mathrm{Re}\langle A_h Y_h, Y_h \rangle_{\mathbb{X}_h} = -h \sum_{jh \in \omega} a_j \left| v_{j+\frac{1}{2}} \right|^2 - \frac{m\alpha |v_0|^2 + a|w_0|^2}{m + a\alpha}, \tag{2.25}$$

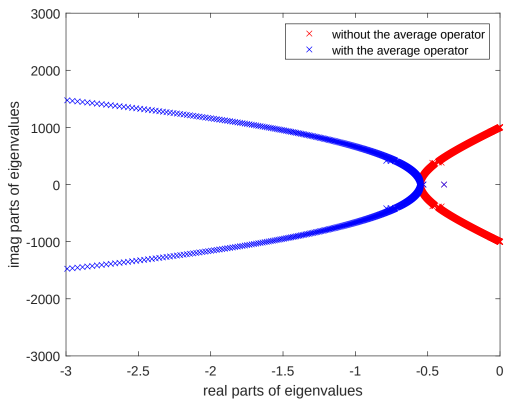

Fig. 2. Eigenvalue distributions with $N = 500$

which implies that A_h is dissipative. $\square$

Moreover, for any $0 < h < 1$, the spectral set $\sigma(A_h)$ of A_h is contained in the open left half-plane of $\mathbb{C}$. This is main content of the following property.

Proposition 2.3. *For all $0 < h < 1$, $i\mathbb{R}$ is contained in the resolvent set $\rho(A_h)$ of A_h.*

Proof. If there exist $\beta \in \mathbb{R}$ and nonzero $Y_h \in \mathbb{X}_h$ such that $i\beta Y_h = A_h Y_h$, then it follows from (2.25) that

$$0 = \mathrm{Re}\,\langle i\beta Y_h,\ Y_h\rangle_{\mathbb{X}_h} = \mathrm{Re}\,\langle A_h Y_h,\ Y_h\rangle_{\mathbb{X}_h} = -h\sum_{jh\in\omega} a_j \left|v_{j+\frac{1}{2}}\right|^2 - \frac{m\alpha|v_0|^2 + a|w_0|^2}{m + a\alpha}.$$

This means that $h\sum_{jh\in\omega} a_j |v_{j+\frac{1}{2}}|^2 = 0$ and $v_0 = w_0 = \eta = 0$ since $\eta = mv_0 - \alpha w_0$. Noting that $i\beta Y = A_h Y$ is equivalent to

$$i\beta v_{j+\frac{1}{2}} = \delta_x w_{j+\frac{1}{2}}, \tag{2.26}$$

$$i\beta w_{j+\frac{1}{2}} = \delta_x v_{j+\frac{1}{2}}. \tag{2.27}$$

Setting $j = 0$ in the above identities, we have

$$\begin{aligned} i\beta h v_1 - 2w_1 &= 0 \\ -2v_1 + i\beta h w_1 &= 0, \end{aligned}$$

It follows from Cramer's rule that $v_1 = w_1 = 0$ since the determinant

$$\begin{vmatrix} i\beta h & -2 \\ -2 & i\beta h \end{vmatrix} = -(\beta h)^2 - 4$$

is nonzero. Similarly, we can repeat this process to prove $v_{j+1} = w_{j+1} = 0$ by solving (2.26)-(2.27) and using $v_j = w_j = 0$. This yields $Y_h = 0$, which is a contradiction and we complete the proof. $\square$

We can now state the following uniform stability standard, which is given in Ref. 1 or Ref. 17 and will be applied in the proof of Theorem 2.3.

Theorem 2.2. *Let $h^* > 0$ and $(S_h(t))$ be a family of semigroups of contraction on the Hilbert space $(\widetilde{\mathbb{X}}_h)$, and let $(\widetilde{A}_h)$ be the corresponding infinitesimal generators. The family $(S_h(t))$ is uniformly exponentially stable if and only if the two following conditions are satisfied:*

(i) For all $h \in (0, h^*)$, $i\mathbb{R}$ is contained in the resolvent set $\rho(\widetilde{A}_h)$ of $(\widetilde{A}_h)$.

(ii) $\sup_{h\in(0,h^*),\beta\in\mathbb{R}} \|(i\beta I - \widetilde{A}_h)^{-1}\|_{L(\widetilde{\mathbb{X}}_h)} < \infty$.

We are in a position to give key result of this section.

Theorem 2.3. *The family $(T_h(t))$ is uniformly exponentially stable, that is to say there exist two constants $M > 0$ and $\omega > 0$ (both independent of $h \in (0,1)$) such*

that

$$\|T_h(t)\|_{L(\mathbb{X}_h)} \le Me^{-\omega t}, \ \forall t \ge 0. \tag{2.28}$$

Proof. The proof is based on Theorem 2.2. Notice first that, for all $h \in (0,1)$, the family $(T_h(t))$ form a family of contraction semigroups (see Proposition 2.2). The fact that the family (A_h) satisfies condition (i) follows from Proposition 2.3. In order to show that the family (A_h) satisfies condition (ii) we use a contradiction argument. If the condition (ii) is false, then there exist $\beta_n \in \mathbb{R}$, $h_n \in (0,1)$, and $Z^n_{h_n} \in \mathbb{X}_{h_n}$ such that

$$\|(i\beta_n I_{h_n} - A_{h_n})^{-1} Z^n_{h_n}\|_{\mathbb{X}_{h_n}} \ge n\|Z^n_{h_n}\|_{\mathbb{X}_{h_n}},$$

in which I_h is the identity operator on $\mathbb{X}_h$, $n = 1, 2, \cdots$, and the norm $\|\cdot\|_{\mathbb{X}_h}$ is induced by the inner product of $\mathbb{X}_h$. Let $(i\beta_n I_{h_n} - A_{h_n})^{-1} Z^n_{h_n} = F^n_{h_n}$ and we have

$$\|F^n_{h_n}\|_{\mathbb{X}_{h_n}} \ge n\|(i\beta_n I_{h_n} - A_{h_n})F^n_{h_n}\|_{\mathbb{X}_{h_n}}.$$

Setting $Y^n_{h_n} = F^n_{h_n}/\|F^n_{h_n}\|_{\mathbb{X}_{h_n}}$, we obtain $\|Y^n_{h_n}\|_{\mathbb{X}_{h_n}} = 1$ and

$$\|U^n\|_{\mathbb{X}_{h_n}} \le n^{-1}, \text{ with } U^n = (i\beta_n I_{h_n} - A_{h_n})Y^n_{h_n}. \tag{2.29}$$

More precisely, for $U^n = (\zeta^n, \xi^n, \theta^n)$ with $\zeta^n = (\zeta^n_0, \zeta^n_1, \cdots, \zeta^n_{N_n}) \in \mathbb{C}^{N_n+1}$, $\xi^n = (\xi^n_1, \xi^n_1, \cdots, \xi^n_{N_n+1}) \in \mathbb{C}^{N_n+1}$, and $\theta^n \in \mathbb{C}$, we have

$$\|U^n\|^2_{\mathbb{X}_{h_n}} = \frac{h_n}{2}\sum_{j=0}^{N_n}\left[\left|\zeta^n_{j+\frac12}\right|^2 + \left|\xi^n_{j+\frac12}\right|^2\right] + \frac{|\theta^n|^2}{2(m+a\alpha)} \le n^{-2}. \tag{2.30}$$

with $\xi^n_0 = \alpha^{-1}(m\zeta^n_0 - \theta^n)$. It follows from Cauchy-Schwartz inequality, (2.29), and (2.25) that

$$\begin{aligned} \mathrm{Re}\,\langle U^n, Y^n_{h_n}\rangle_{\mathbb{X}_{h_n}} &= -\mathrm{Re}\,\langle A_{h_n}Y^n_{h_n}, Y^n_{h_n}\rangle_{\mathbb{X}_{h_n}} \\ = h_n \sum_{jh\in\omega} a_j \left|v^n_{j+\frac12}\right|^2 &+ \frac{m\alpha|v^n_0|^2 + a|w^n_0|^2}{m+a\alpha} \le n^{-1}. \end{aligned} \tag{2.31}$$

Therefore, by using inequalities (2.30) and (2.31), we have

$$|\zeta^n_{j+\frac12}|^2 = O(n^{-2}), \ j = 0, 1, \cdots, N_n \tag{2.32}$$

$$|\xi^n_{j+\frac12}|^2 = O(n^{-2}), \ j = 0, 1, \cdots, N_n, \tag{2.33}$$

$$\sum_{jh\in\omega} a_j \left|v^n_{j+\frac12}\right|^2 = O(n^{-1}), \tag{2.34}$$

$$|v^n_0|^2 = O(n^{-1}), \tag{2.35}$$

$$|w^n_0|^2 = O(n^{-1}). \tag{2.36}$$

The notation $|w^n_0|^2 = O(n^{-1})$ in (2.36) means that there is a positive constant C (independent of n) such that $|w^n_0|^2 \le Cn^{-1}$. The other notation $O(\cdot)$ in (2.32)-(2.35) has the same meaning.

Furthermore, combing $B_{h_n}U^n$ with $U^n = (i\beta_n I_{h_n} - A_{h_n})Y^n_{h_n}$, we obtain

$$i\beta_n v^n_{j+\frac{1}{2}} - \delta_x w^n_{j+\frac{1}{2}} + a_j v^n_{j+\frac{1}{2}} = \zeta^n_{j+\frac{1}{2}}, \tag{2.37}$$

$$i\beta_n w^n_{j+\frac{1}{2}} - \delta_x v^n_{j+\frac{1}{2}} = \xi^n_{j+\frac{1}{2}}. \tag{2.38}$$

By rearranging (2.37) and (2.38), we derive

$$i\beta_n h_n v^n_{j+\frac{1}{2}} - 2w^n_{j+\frac{1}{2}} = -2w^n_j + p^n_j, \tag{2.39}$$

$$-2v^n_{j+\frac{1}{2}} + i\beta_n h_n w^n_{j+\frac{1}{2}} = -2v^n_j + h_n \xi^n_{j+\frac{1}{2}}, \tag{2.40}$$

in which

$$p^n_j = h_n \left[a_j v^n_{j+\frac{1}{2}} + \zeta^n_{j+\frac{1}{2}} \right]. \tag{2.41}$$

Set $j = 0$ in (2.39) and (2.40) and solve the resulting linear equation to arrive at

$$\begin{pmatrix} v^n_{0+\frac{1}{2}} \\ w^n_{0+\frac{1}{2}} \end{pmatrix} = \frac{-1}{4+\beta_n^2 h_n^2} \begin{pmatrix} i\beta_n h_n & 2 \\ 2 & i\beta_n h_n \end{pmatrix} \begin{pmatrix} -2w^n_0 + p^n_0 \\ -2v^n_0 + h_n \xi^n_{0+\frac{1}{2}} \end{pmatrix},$$

and which implies

$$|v^n_{0+\frac{1}{2}}|^2 = O(n^{-1}), \quad |w^n_{0+\frac{1}{2}}|^2 = O(n^{-1}), \tag{2.42}$$

by (2.32)-(2.36), (2.41) and some basic calculations. Now (2.35)-(2.36) and (2.42) together yields

$$|v^n_1|^2 = \left| 2v^n_{0+\frac{1}{2}} - v^n_0 \right|^2 = O(n^{-1}),$$

$$|w^n_1|^2 = \left| 2w^n_{0+\frac{1}{2}} - w^n_0 \right|^2 = O(n^{-1}),$$

By letting $j = 1$ in (2.39) and (2.40), we can obtain

$$|v^n_{1+\frac{1}{2}}|^2 = O(n^{-1}), \quad |w^n_{1+\frac{1}{2}}|^2 = O(n^{-1}),$$

$$|v^n_2|^2 = O(n^{-1}), \qquad |w^n_2|^2 = O(n^{-1}),$$

by using the same method as above. Gradually, we can derive that

$$|v^n_{j+\frac{1}{2}}|^2 = O(n^{-1}), \quad |w^n_{j+\frac{1}{2}}|^2 = O(n^{-1}),$$

hold for all $j = 0, 1, \cdots, N_n$ and even more

$$\frac{h_n}{2} \sum_{j=0}^{N_n} \left[\left| v^n_{j+\frac{1}{2}} \right|^2 + \left| w^n_{j+\frac{1}{2}} \right|^2 \right] = O(n^{-1}) \tag{2.43}$$

since $h_n(N_n + 1) = 1$. It is easy to see from (2.35)-(2.36) that

$$|\eta^n|^2 = |m v^n_0 - \alpha w^n_0|^2 = O(n^{-1}). \tag{2.44}$$

Finally, (2.43) and (2.44) imply that $\|Y^n_{h_n}\|_{\mathbb{X}_{h_n}} = O(n^{-1})$ for all positive integers n, which is a contradiction to $\|Y^n_{h_n}\|_{\mathbb{X}_{h_n}} = 1$. The proof of the Theorem 2.3 is complete. □

2.3. *Convergence analysis*

The goal of this subsection is to show the approximating generators A_h, on the spaces $\mathbb{X}_h$, such that the C_0-semigroups $T_h(\cdot)$ generated by A_h, approximate $T(\cdot)$ in some sense. We will need the following assumptions: for every $h \in (0,1)$ there exist bounded linear operators $P_h : \mathbb{X} \to \mathbb{X}_h$ and $E_h : \mathbb{X}_h \to \mathbb{X}$ satisfying

(A_1) There exist two positive constants M_1 and M_2 such that $\|E_h\|_{\mathbb{X}} \le M_1$ and $\|P_h\|_{\mathbb{X}} \le M_2$,

(A_2) $\|E_hP_hY - Y\|_{\mathbb{X}} \to 0$ as $h \to 0$ for all $Y \in \mathbb{X}$,

(A_3) $P_hE_h = I_h$.

Theorem 2.4. *[Trotter-Kato[10]] Assume that $(A1)$ and $(A3)$ are satisfied. Then the following statements are equivalent:*

(a) There exists a $\lambda_0 \in \rho(A) \cap \bigcap_{n=1}^{\infty} \rho(A_h)$ such that, for all $Y \in \mathbb{X}$,

$$\|E_h(\lambda_0 I_h - A_h)^{-1}P_hY - (\lambda_0 I - A)^{-1}Y\|_{\mathbb{X}} \to 0, \ \text{as } h \to 0.$$

(b) For every $Y \in \mathbb{X}$ and $t \ge 0$,

$$\|E_hT_h(t)P_hY - T(t)Y\|_{\mathbb{X}} \to 0, \ \text{as } h \to 0$$

uniformly on bounded t-intervals.

However, in order to prove Theorem 2.4 one faces several major difficulties. The most difficult one is the direct verification of the consistency property (a). It involves computation of the resolvents $(\lambda_0 I_h - A_h)^{-1}$, which in general is almost impossible. Therefore, the following property is useful. With the aide of it one can replace (a) by a condition involving convergence of the operators A_h to A in some sense of Ref. 10.

Proposition 2.4. *Let the assumptions of Theorem 2.4 be satisfied. Then statement (a) of Theorem 2.4 is equivalent to $(A2)$ and the following two statements:*

$(C1)$ There exists a subset $D \subseteq D(A)$ such that $\overline{D} = \mathbb{X}$ and $(\lambda_0 I - A)^{-1}D = \mathbb{X}$ for a $\lambda_0 > \omega$.

$(C2)$ For all $Y \in D$ there exists a sequence $(\overline{Y}_h)$ with $\overline{Y}_h \in D(A_h)$ such that

$$\lim_{h\to 0} E_h\overline{Y}_h = Y, \quad \lim_{h\to 0} E_hA_h\overline{Y}_h = AY. \tag{2.45}$$

Proof of Theorem 2.4 Let χ_S be the characteristic function of a set S and define the matrix $F_h(x)$ consisting of characteristic functions $\chi_{(x_j,x_{j+1}]}$ $(j = 0, 1, \cdots, N)$ by

$$F_h(x) = \begin{pmatrix} \chi_{(x_0,x_1]}, \chi_{(x_1,x_2]}, \cdots, \chi_{(x_N,x_{N+1}]} \\ \chi_{(x_0,x_1]}, \chi_{(x_1,x_2]}, \cdots, \chi_{(x_N,x_{N+1}]} \\ \vdots \\ \chi_{(x_0,x_1]}, \chi_{(x_1,x_2]}, \cdots, \chi_{(x_N,x_{N+1}]} \end{pmatrix}.$$

For any $Y_h = (v, w, \eta) \in \mathbb{X}_h$ with $v = (v_0, v_1, \cdots, v_N)$ and $w = (w_1, w_2, \cdots, w_{N+1})$, set $w_0 = \alpha^{-1}(mv_0 - \eta)$ and define the extension operators $E_h : \mathbb{X}_h \to \mathbb{X}$:

$$E_h Y_h = \mathrm{diag}(F_h(x), F_h(x), 1) B_h Y_h = \begin{pmatrix} \sum_{j=0}^N (v_{j+\frac{1}{2}}) \chi_{(x_j, x_{j+1}]} \\ \sum_{j=0}^N (w_{j+\frac{1}{2}}) \chi_{(x_j, x_{j+1}]} \\ \eta \end{pmatrix}. \tag{2.46}$$

Choose the dense subset $D \triangleq D(A) \bigcap (C^2[0,1])^2 \times \mathbb{C}$ of $\mathbb{X}$. For any $Y = (v(\cdot), w(\cdot), \eta) \in D$, set $\overline{v} = (v(x_0),\ v(x_1),\ \cdots,\ v(x_N))^*$, $\overline{w} = (w(x_1),\ w(x_0),\ \cdots,\ w(x_{N+1}))^*$, and $\overline{Y}_h = (\overline{v}, \overline{w}, \eta)^*$. By using $v(1) = v(x_{N+1}) = 0$ and $w(x_0) = \alpha^{-1}[mv(x_0) - \eta]$ (it comes from the definition of the operator A), it is easy to see that

$$E_h \overline{Y}_h = \begin{pmatrix} \sum_{j=0}^N \left(\frac{v(x_{j+1}) + v(x_j)}{2} \right) \chi_{[x_j, x_{j+1}]} \\ \sum_{j=0}^N \left(\frac{w(x_{j+1}) + w(x_j)}{2} \right) \chi_{[x_j, x_{j+1}]} \\ \eta \end{pmatrix},$$

and

$$E_h A_h \overline{Y}_h = \mathrm{diag}(F_h(x), F_h(x), 1) M_h Y_h = \begin{pmatrix} \sum_{j=0}^N \left(\frac{w(x_{k+1}) - w(x_k)}{h} \right) \chi_{[x_j, x_{j+1}]} \\ \sum_{i=0}^N \left(\frac{v(x_{j+1}) - v(x_j)}{h} \right) \chi_{[x_j, x_{j+1}]} \\ mv(x_0) - \alpha w(x_0) \end{pmatrix}.$$

By using the same method of Section 4 in Ref. 10, we know that the conditions (C_1) and (C_2) of Proposition 2.4 hold.

To prove (b) of Theorem 2.4 and give the main result of this subsection, we need further to verify (A_1)-(A_3). In view of (A_3) and (2.46), construct the projecting operators $P_h : X \to X_h$

$$P_h Y = \begin{pmatrix} I^0 v(x) \\ I^1 w(x) \\ \eta \end{pmatrix},\ I^l f_l(x) = 2 \begin{pmatrix} I_l^l f_l(x) \\ I_{1+l}^l f_l(x) \\ \vdots \\ I_{N+l}^l f_l(x) \end{pmatrix},$$

for $Y \in \mathbb{X}_h$, $l = 0, 1,\ f_0(x) = v(x),\ f_1(x) = w(x)$. $I_j^0 v(x)$ and $I_{j+1}^1 w(x)$ are defined by

$$\begin{aligned} &I_N^0 v(x) = h^{-1} \int_{x_N}^{x_{N+1}} v(x) dx, \\ &I_j^0 v(x) = h^{-1} \int_{x_j}^{x_{j+1}} v(x) dx - I_{j+1}^0 v(x) \\ &I_1^1 w(x) = h^{-1} \int_{x_0}^{x_1} w(x) dx - \alpha^{-1} [I_0^0 v(x) - \eta], \\ &I_{j+1}^1 w(x) = h^{-1} \int_{x_{j+1}}^{x_{j+2}} w(x) dx - I_j^1 w(x). \end{aligned}$$

In light of the definitions of E_h and P_h, it is easy to see that they satisfy (A_1) and (A_3). To prove that E_N and P_N satisfy $(A2)$, we firstly assume $Y = (v(x), w(x), \eta) \in$

$D(A)$. With this assumption we have

$$E_hP_hY = h^{-1}\begin{pmatrix} \sum_{j=0}^{N}\int_{x_j}^{x_{j+1}} v(x)dx\chi_{[x_j,x_{j+1}]} \\ \sum_{j=0}^{N}\int_{x_j}^{x_{j+1}} w(x)dx\chi_{[x_j,x_{j+1}]} \\ h\eta \end{pmatrix}$$

and

$$E_hP_hY - Y = \begin{pmatrix} \sum_{j=0}^{N}[v(\theta_j^v) - v(x)]\chi_{[x_j,x_{j+1}]} \\ \sum_{j=0}^{N}[v(\theta_j^w) - w]\chi_{[x_j,x_{j+1}]} \\ 0 \end{pmatrix} \to 0, \text{ as } h \to 0,$$

here θ_j^v and θ_j^w are chosen such that $\int_{x_j}^{x_{j+1}} v(x)\mathrm{d}x = v(\theta_j^v)h$ and $\int_{x_j}^{x_{j+1}} w(x)\mathrm{d}x = w(\theta_j^w)h$ when the mean value theorem is applied, and the continuity of $v(x)$ and $w(x)$ on the internal $[0,1]$ is applied in the last step. Thus combing this result and the density of $D(A)$ in the state space H, we obtain (A_2) and complete the proof of Theorem 2.4. □

Using Theorem 2.3 and Theorem 2.4, we can give the exponential stability of (2.1)-(2.5).

Theorem 2.5. *Let $Y(0) \in \mathbb{X}$ be defined by (2.6) and set $Y_h^0 := P_hY(0)$ be the initial data of (2.12)-(2.17), then Y_h^0 convergent to the initial value $Y(0)$ of (2.1)-(2.5) in the sense of*

$$E_hY_h^0 \to Y(0), \ \textit{as } h \to 0,$$

and the semigroup $T(t)$ is exponentially stable, i.e.,

$$\|T(t)Y(0)\|_{\mathbb{X}} \le M'e^{-\omega t}\|Y(0)\|_{\mathbb{X}}, \quad \forall\ Y(0) \in \mathbb{X},\ t \ge 0,$$

for some constant M' and ω is the same as in Theorem 2.3.

Proof. As $h \to 0$, $E_hY_h^0$ convergent to $Y(0)$ since E_h and P_h satisfy (A_2). For any $\varepsilon > 0$, (b) of Theorem 2.4 implies that there exists $h_\varepsilon \in (0,1)$ such that

$$\|E_hT_h(t)P_hY(0) - T(t)Y(0)\|_{\mathbb{X}} < \varepsilon$$

for all $h < h_\varepsilon$. However, it follows from (2.28) and (A_1) that

$$\|E_hT_h(t)P_hY(0)\|_{\mathbb{X}} \le M'e^{-\omega t}\|Y(0)\|_{\mathbb{X}}, \quad \text{for } h < h_\varepsilon,$$

with $M' = MM_1M_2$. Therefore, combining the above two inequalities, we obtain

$$\|T(t)Y(0)\|_{\mathbb{X}} \le M'e^{-\omega t} + \varepsilon.$$

By letting $\varepsilon \to 0$, we derive that $T(t)$ is exponentially stable. □

3. The hybrid system with nonlinear feedback

In this section, we cope with the hybrid system (1.1) and (1.7)-(1.8)

$$\begin{aligned}&\partial_{tt}u(x,t)=\partial_{xx}u(x,t)+a(x)u_t(x,t),\\&m\partial_{tt}u(0,t)-\partial_x u(0,t)=F(u_t(0,t)),\\&u_x(1,t)=-g(u_t(1,t)),\end{aligned}$$

Using the auxiliary functions $v(x,t)=u_t(x,t)$ and $w(x,t)=u_x(x,t)$, the above hybrid system can be rewritten as

$$\partial_t v(x,t)=\partial_x w(x,t)-a(x)v(t,x), \tag{3.1}$$

$$\partial_t w(x,t)=\partial_x v(x,t), \tag{3.2}$$

$$m\partial_t v(0,t)=F(v(0,t))+w(0,t), \tag{3.3}$$

$$w(1,t)=-g(v(1,t)), \tag{3.4}$$

$$v(x,0)=v^0(x),\quad w(x,0)=w^0(x), \tag{3.5}$$

in which $v^0(x)$ and $w^0(x)$ are given in the last section. $a(x)$, $g(s)$ and $F(s)$ are scalar functions satisfying the following hypothesises[26]:

- (H_1) $a(x)\in C^1[0,1]$ and $|a(x)|\le\alpha_1$ with $\alpha_1\in(0,1)$ and $m\alpha_1<4$;
- (H_2) $g(s)$ is continuous, nondecreasing, and $g(0)=0$;
- (H_3) $F(s)$ is globally q-Lipschitz continuous, and $F(0)=0$;
- (H_4) $\alpha_2|s|\le|g(s)|\le\alpha_3|s|$, for all $s\in\mathbb{C}$ and some positive real numbers α_2 and α_3;
- (H_5) The inequality $\alpha_2/(1+\alpha_3^2)-q>\alpha_1(4-m\alpha_1)$ holds.

We refer the reader to Ref. 26 for well-posedness properties of the closed-loop system (3.1)-(3.5).

3.1. *UES of (3.1)-(3.5)*

Under the above assumptions, we analyze the UES of (3.1)-(3.5). For this purpose, we use the average central-difference scheme of last section to discretize the spacial derivative of (3.1)-(3.5) and obtain

$$v'_{j+\frac{1}{2}}(t)=\delta_x w_{j+\frac{1}{2}}(t)-a_j v_{j+\frac{1}{2}}(t), \tag{3.6}$$

$$w'_{j+\frac{1}{2}}(t)=\delta_x v_{j+\frac{1}{2}}(t), \tag{3.7}$$

$$mv'_0(t)=F(v_0(t))+w_0(t), \tag{3.8}$$

$$w_{N+1}(t)=-g(v_{N+1}(t)), \tag{3.9}$$

$$v_k(0)=v_k^0,\quad w_k(0)=w_k^0, \tag{3.10}$$

where $v_j(t)$, $w_j(t)$, a_k, v_k^0 and w_k^0 have the same meaning as in the last section. For simplicity, we drop for readability the coordinate t of $v_j(t)$ and $w_j(t)$ from now on.

The energy functional, which is denoted by $E_{nh}(t)$, of (3.6)-(3.10) is defined by

$$E_{nh}(t) = \frac{h}{2}\sum_{j=0}^{N}\left[\left|v_{j+\frac{1}{2}}\right|^2 + \left|w_{j+\frac{1}{2}}\right|^2\right] + \frac{m}{2}|v_0|^2, \tag{3.11}$$

which satisfies similar result as proposition 2.1.

Proposition 3.1. *The discrete energy functional $E_{nh}(t)$ satisfies*

$$E'_{nh}(t) = -h\sum_{jh\in\omega} a_j|v_{j+\frac{1}{2}}|^2 - v_{N+1}g(v_{N+1}) + v_0F(v_0). \tag{3.12}$$

Proof. Multiplying (3.6), (3.7), and (3.8) by $hv_{j+\frac{1}{2}}$, $hw_{j+\frac{1}{2}}$, and v_0 respectively and summing for j, we derive

$$E'_{nh}(t) = -h\sum_{jh\in\omega} a_j|v_{j+\frac{1}{2}}|^2 + h\sum_{j=0}^{N}[v_{j+\frac{1}{2}}\delta_x w_{j+\frac{1}{2}} + w_{j+\frac{1}{2}}\delta_x v_{j+\frac{1}{2}}]. \tag{3.13}$$

However, by using (2.23), we have

$$h\sum_{j=0}^{N}[v_{j+\frac{1}{2}}\delta_x w_{j+\frac{1}{2}} + w_{j+\frac{1}{2}}\delta_x v_{j+\frac{1}{2}}] = v_{N+1}w_{N+1} - v_0w_0. \tag{3.14}$$

Thus, (3.8)-(3.9) and (3.13)-(3.14) implies that (3.12) holds. □

Remark 3.1. Because no prior knowledge on the signs of the functions a and F, the energy functional $E_{nh}(t)$ is no longer non-increasing along the trajectories of the system. This means that $F(s)$ and $a(x)$ can be model destabilizing boundary and interior anti-damping phenomenons.

To present main result of this section, we introduce two useful lemmas. The first one comes from Lemma 2.2 of Ref. 13 or Lemma 2.1 of Ref. 30. The second one is discrete counterpart of (57) of Ref. 26 or (A3) of Ref. 5.

Lemma 3.1. *Let $\{u_i\}_{i=0}^{N+1}$, $\{v_i\}_{i=0}^{N+1}$ and $\{w_i\}_{i=0}^{N+1}$ be sequences consisting of real numbers, then we have*

$$\begin{aligned}
&\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}+v_i)(w_{i+1}+w_i) + \frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}-v_i)(w_{i+1}-w_i)\\
&+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}-v_i)(w_{i+1}+w_i) + \frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}+v_i)(w_{i+1}-w_i)\\
&= u_{N+1}v_{N+1}w_{N+1} - u_0v_0w_0
\end{aligned}$$

Lemma 3.2. *Let $\rho(x)$ be affine, positive, and increasing, $\rho' = \rho'(x)$, $\rho_j = \rho(x_j)$, and $L_h(t)$ be the Lyapunov functional which is defined by*

$$L_h(t) = E_{nh}(t) + h\sum_{j=0}^{N}\rho_{j+\frac{1}{2}}v_{j+\frac{1}{2}}w_{j+\frac{1}{2}}, \tag{3.15}$$

then for any $T>0$ we have

$$
\begin{aligned}
& L_h(t)|_0^T \\
\le & -(\rho'-2\alpha_1)\int_0^T E_{nh}(t)dt-\int_0^T v_{N+1}g(v_{N+1})dt+2\rho_{N+1}\int_0^T |v_{N+1}|^2+|w_{N+1}|^2dt \\
& +\int_0^T v_0F(v_0)dt-(2\rho_0+m\alpha_1/2+m\alpha_1\rho'/2)\int_0^T |v_0|^2dt.
\end{aligned}
\tag{3.16}
$$

Proof. It is easy to know

$$
E_{nh}(t)|_0^T \le \int_0^T \alpha_1 E_{nh}(t)dt-\int_0^T \frac{m\alpha_1}{2}|v_0|^2+v_{N+1}g(v_{N+1})-v_0F(v_0)dt \tag{3.17}
$$

holds by (3.11)-(3.12) and (H_1). Just as in the proof of Proposition 3.1, we multiply (3.6) and (3.7) by $h\rho_{j+\frac12}w_{j+\frac12}$ and $h\rho_{j+\frac12}v_{j+\frac12}$ and obtain

$$
\begin{aligned}
& h\sum_{j=0}^{N}\rho_{j+\frac12}v_{j+\frac12}w_{j+\frac12}\Bigg|_0^T \\
= & -\int_0^T h\sum_{jh\in\omega} a_jv_{j+\frac12}w_{j+\frac12}dt+\int_0^T h\sum_{j=0}^{N}\rho_{j+\frac12}\left[v_{j+\frac12}(t)\delta_xv_{j+\frac12}+\delta_xw_{j+\frac12}w_{j+\frac12}\right]dt.
\end{aligned}
\tag{3.18}
$$

Applying Lemma 3.1 for the second term in the right-hand side of (3.18), we obtain

$$
\begin{aligned}
& h\sum_{j=0}^{N}\rho_{j+\frac12}[v_{j+\frac12}\delta_xv_{j+\frac12}+\delta_xw_{j+\frac12}w_{j+\frac12}] \\
= & -\frac{h}{2}\sum_{j=0}^{N}\delta_x\rho_{j+\frac12}[|v_{j+\frac12}(t)|^2+|w_{j+\frac12}|^2]+2\rho_{N+1}[|v_{N+1}|^2+|w_{N+1}|^2] \\
& -2\rho_0[|v_0|^2+|w_0|^2]-\frac{h^3}{2}\sum_{j=0}^{N}\delta_x\rho_{j+\frac12}[|\delta_xv_{j+\frac12}|^2+|\delta_xw_{j+\frac12}|^2].
\end{aligned}
\tag{3.19}
$$

Then, combining (3.18) and (3.19) yields

$$
\begin{aligned}
& h\sum_{j=0}^{N}\rho_{j+\frac12}v_{j+\frac12}w_{j+\frac12}\Bigg|_0^T \le -\int_0^T(\rho'-\alpha_1)E_{nh}(t)dt \\
& +2\rho_{N+1}\int_0^T |v_{N+1}|^2+|w_{N+1}|^2dt-(2\rho_0+m\alpha_1\rho'/2)\int_0^T |v_0|^2dt,
\end{aligned}
\tag{3.20}
$$

where Cauchy-Schwartz inequality is used for the first term of the right-hand side of (3.18), $\delta_x\rho_{j+\frac12}=\rho'>0$, (3.11), and (H_1) are applied in (3.19). At last, plugging (3.20) into (3.17), we derive (3.16) and finish the proof of this lemma. □

With the aid of Lemma 3.2, we can derive UES of (3.6)-(3.10) under the hypothesises (H_1)-(H_5).

Theorem 3.1. *Assume that (H_1)-(H_5) hold, the discrete systems (3.6)-(3.10) are uniformly exponentially stable with respect to the energy $E_{nh}(t)$, i.e., there exist two positive constants $\mathcal{M}$ and ω_1 (independent if h and solutions) such that*

$$E_{nh}(t) \le \mathcal{M} e^{-\omega_1 t} E_{nh}(0), \text{ for all } t \ge 0. \tag{3.21}$$

Proof. Firstly, from (3.15) we know

$$(1-\alpha_1)E_{nh}(t) \le L_{nh}(t) \le (1+\alpha_1)E_{nh}(t) \tag{3.22}$$

since

$$|h\sum_{j=0}^{N} \delta^{\frac{1}{2}}\rho_j \delta^{\frac{1}{2}} v_j \delta^{\frac{1}{2}} w_j| \le \alpha_1 E_{nh}(t).$$

Secondly, (H_3) implies $sF(s) \le q|s|^2$ for all $s \in \mathbb{R}$ and

$$\int_0^T v_0 F(v_0)dt - (2\rho_0 + m\alpha_1/2 + m\alpha_1\rho'/2)\int_0^T |v_0|^2 dt \le -m\alpha_1/2 \int_0^T |v_0|^2 dt, \tag{3.23}$$

where $2\rho_0 + m\alpha_1\rho'/2 = q$ is assumed for a moment. By using (3.9) and (H_4), we see that

$$2\rho_{N+1}\int_0^T |v_{N+1}|^2 + |w_{N+1}|^2 dt \le 2\rho_{N+1}(1+\alpha_3^2)\int_0^T |v_{N+1}|^2 dt. \tag{3.24}$$

Moreover, combing (H_2) and (H_4) yields

$$-\int_0^T v_{N+1} g(v_{N+1})dt \le -\alpha_2 \int_0^T |v_{N+1}|^2 dt. \tag{3.25}$$

Finally, assuming $\rho_{N+1} = \alpha_2/2(1+\alpha_3^2)$, we can know that the affine function $\rho(x)$ is uniquely defined by noticing $\rho_{N+1} = \rho_0 + \rho'$ and collecting above relations on ρ

$$\begin{aligned} 2\rho_0 + m\alpha_1\rho'/2 &= q, \\ \rho_0 + \rho' &= \frac{\alpha_2}{2(1+\alpha_3^2)}. \end{aligned}$$

ρ_0 and ρ' are uniquely existed since $m\alpha_1 < 4$ and $\rho(x)$ is unique defined by ρ_0 and ρ' and indeed increasing with $\rho' = 2[\alpha_2/(1+\alpha_3^2) - q]/(4 - m\alpha_1) > 0$ (see H_5). Now, combing (3.16) and (3.22)-(3.25), we obtain

$$L_{nh}(t)|_0^T \le -\omega_1 \int_0^T L_{nh}(t)dt, \text{ for all } T > 0.$$

where

$$\omega_1 = \frac{2[\alpha_2/(1+\alpha_3^2) - q]/(4 - m\alpha_1) - 2\alpha_1}{1-\alpha_1}.$$

By applying (H_1), (H_5), and Grönwall lemma, we obtain the UES of (3.21). The proof of the Theorem 3.1 is complete. □

Remark 3.2. We can perform convergence analysis for the discrete system (3.6)-(3.10) by using the methods of Ref. 13, Ref. 5 or Ref. 24. But we delete it since the methods are very standard and the dynamical boundary conditions and nonlinear feedback do not effect the convergence. Moreover, we know that the discrete energy functional E_{nh} also is convergent to the energy functional of (3.1)-(3.5) by similar result Theorem 5.1 of Ref. 13. Therefore, the nonlinear counterpart of Theorem 2.5 is still right by the same analysis. That is to say (3.1)-(3.5) is exponentially stable under the hypothesis of (H_1)-(H_5).

3.2. *Numerical simulations*

Firstly, we rearrange (3.6)-(3.10) into vectorial formulation. The unknown of (3.6)-(3.10) is $Y_{nh}(t) = (v_h(t), v_{N+1}(t), w_h(t))^*$. $v_h(t)$, $w_h(t)$, B_{N+1}, M_{N+1}, e_1, and Λ_a are the same as those of subsection 2.2. Moreover, set $a(x) = 1/10\cos(\pi x)$, $q = 1/10$, $\alpha_2 = 1$, $\alpha_3 = 2$, $e_{N+1} = (0, 0, \cdots, 0, 1) \in \mathbb{C}^{(N+1)}$ to be row vector.

Secondly, in light of hypothesises (H_2)-(H_5), for $s \in \mathbb{C}$ we choose nonlinear functions

$$F(s) = \begin{cases} \frac{1}{s+q^{-1/2}} - \sqrt{q} + qs, & s \geq 0, \\ -\frac{1}{-s+q^{-1/2}} + \sqrt{q} + qs, & s < 0, \end{cases}$$

and

$$g(s) = \begin{cases} -1 + 2s + \frac{1}{s+1}, & s \geq 0, \\ 1 + 2s - \frac{1}{-s+1}, & s < 0. \end{cases} \tag{3.26}$$

$F(s)$ clearly satisfies (H_3) due to

$$\frac{\mathrm{d}}{\mathrm{d}s} F(s) = q - \frac{1}{\left(|s| + q^{-1/2}\right)^2} \leq q, \tag{3.27}$$

which implies that $F(s)$ is globally q-Lipschitz continuous. $g(s)$ is obviously continuous and $g(0) = 0$. Moreover, $g(s)$ satisfies (H_2) and (H_4) since

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s) = 2 - \frac{1}{\left(|s| + 1\right)^2}. \tag{3.28}$$

At last, we choose $m = 39$ and it is easy to verify that $\alpha_1 = 1/10$, $q = 1/10$, $\alpha_2 = 1$, $\alpha_3 = 2$ and m satisfy (H_5). Thus, using the mean value Theorem, $F(0) = 0$, and $g(0) = 0$, we replace $F(v_0(t))$ and $g(v_{N+1}(t))$ by

$$\frac{\mathrm{d}}{\mathrm{d}s} F(v_0(t)) \text{ and } \frac{\mathrm{d}}{\mathrm{d}s} g(v_{N+1}(t)),$$

respectively. Therefore, we obtain the quasi-linearized system of (3.6)-(3.9)

$$v'_{j+\frac{1}{2}}(t) = \delta_x w_{j+\frac{1}{2}}(t) - a_j v_j(t + \frac{1}{2}), \tag{3.29}$$

$$w'_{j+\frac{1}{2}}(t) = \delta_x v_{j+\frac{1}{2}}(t), \tag{3.30}$$

$$v'_0(t) = \frac{\mathrm{d}}{\mathrm{d}s} F(v_0(t)) v_0(t) + w_0(t), \tag{3.31}$$

$$w_{N+1}(t) = -\frac{\mathrm{d}}{\mathrm{d}s} g(v_{N+1}(t)) v_{N+1}(t). \tag{3.32}$$

In order to perform numerical simulations and give ideal approximating of

$$\frac{\mathrm{d}}{\mathrm{d}s} F(v_0(t)) \text{ and } \frac{\mathrm{d}}{\mathrm{d}s} g(v_{N+1}(t)),$$

we replace s by the positive random number ranging from 0 to 10000, which is denoted by r, in (3.27) and (3.28). We define the matrixes B_{nh} and M_{nh} by

$$B_{nh} = \frac{1}{2} \begin{pmatrix} B_{N+1} & e^*_{N+1} & 0 \\ 2e_1 & 0 & 0 \\ 0 & \left(\frac{1}{(r+1)^2 - 2}\right) e^*_{N+1} & B_{N+1} \end{pmatrix}$$

and

$$M_{nh} = \frac{1}{h} \begin{pmatrix} \frac{h}{2} B_{N+1} \Lambda_a & \left(-\frac{h}{2} a_{N+1} - 2 + \frac{1}{(r+1)^2}\right) e^*_{N+1} & -M_{N+1} \\ qhe_1 - \frac{h}{\left(r+q^{-1/2}\right)^2} e_1 & 0 & 0 \\ -M^*_{N+1} & e^*_{N+1} & 0 \end{pmatrix}.$$

Because B_{nh} is evidently invertible, the linearized system (3.29)-(3.32) are then obtained

$$B_{nh} Y'_{nh}(t) = M_{nh} Y_{nh}(t) \Leftrightarrow Y'_{nh}(t) = A_{nh} Y_h(t), \text{ with } A_{nh} := B^{-1}_{nh} M_{nh}. \tag{3.33}$$

Similarly as in subsection 2.2, one replace B_{nh} by identity operator and the linearized classical finite difference scheme of (3.1)-(3.4) is easily restored from (3.33), i.e.,

$$\dot{Y}_{nh}(t) = M_{nh} Y_{nh}(t).$$

Here we explain the significance of the discrete scheme (3.33) by plotting two kinds of figures. Figure 3 gives the maximal real parts of A_{nh} and M_{nh} for $N = 40 : 5 : 400$. Figure 4 depicts the distributions of the eigenvalues of M_{nh} with $N = 500$. We see that the real parts of the eigenvalues of M_{nh} approach to zero form above and those of A_{nh} approach to a negative number from these figures. Numerical simulation results show that the classical finite difference scheme of (3.1)-(3.4) is not uniformly exponentially stable and even not stable. This conclusion is consistent with it's non-dissipativity or anti-damping.

4. Concluding remarks

The exponential stabilities of two hybrid systems are investigated from different viewpoint as before. More precisely, semi-discretized systems and their UESs are first derived. The exponential stability of the hybrid system follows from the convergence of numerical approximating scheme and the result of UES. Looking back at the Subsection 2.2, one may find out that the linear feedback $-au_t(0,t)+\alpha u_{xt}(0,t)$ play key role for UES of discrete hybrid system with linear feedback. However, the local viscosity term $a(x)u_t(x,t)$ is trivial to the main result. This phenomenon also appears in the hybrid system with nonlinear feedback of Section 3. This manifest that velocity stabilizer or velocity plugging angular velocity stabilizer is powerful. We adopt two approaches, namely the frequency domain method and discrete

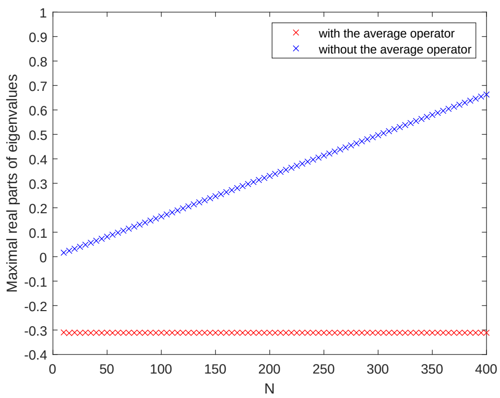

Fig. 3. Maximal real parts of eigenvalues

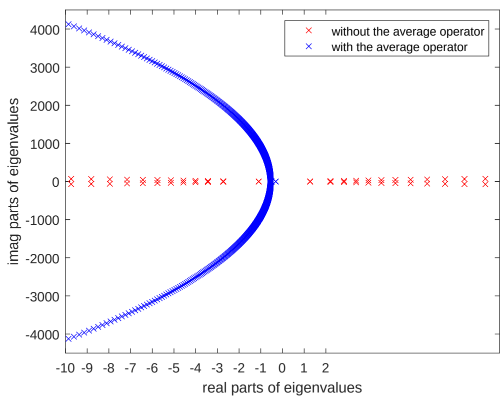

Fig. 4. Eigenvalue distributions with $N = 500$

multiplier method, to verify the UES of corresponding system. In the future, it is interesting to apply frequency domain verification method developed in this paper to other systems with boundary control. Moreover, we only consider one case of the nonlinear hybrid system of Ref. 26, it is worth investigating other cases in the further research.

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